MATH 4613 GALOIS THEORY — PROF. PAUL BAILEY — Problem Set 1 — September 11, 2006 — Name:

Due Date: Friday, September 22,2006.

Write your solutions neatly on separate pieces of paper and attach this sheet to the front. Just do your best and turn it in on the due date; no late papers this time. Remember that the first step to failure is giving up.

**Problem 1.** Given a circle, construct a concentric circle with double the area. Describe each step, and draw all steps with a straight-edge and compass, labeling each point significant for the construction. Explain why your construction works. You may use propositions one through five in the notes (when constructing a midpoint or a perpendicular via these propositions, you may use a ruler or protractor to get a more accurate picture). Recall that "given a circle" means that you are given the center and one point on the circumference.

**Problem 2.** Let

$$f(x) = x^5 + 3x^3 + 2x^2 + 7,$$

and let $\beta \in \mathbb{C}$ be a root of $f$. Find rational numbers $a_0, a_1, a_2, a_3, a_4 \in \mathbb{Q}$ such that

$$\beta^{-1} = a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3 + a_4\beta^4.$$

(Hint: set $f(\beta) = 0$, subtract 7 from both sides, factor out $\beta$, divide by $-7$, and say why this proves what you want).

**Problem 3.** Consider cube roots of complex numbers.

**(a)** Find all complex cube roots of $1 + i$. Explain your reasoning.

**(b)** Does there exist a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^3 = z$? Explain your reasoning.

**Problem 4.** Let $\mathbb{Q}[x]$ denote the set of polynomials with coefficients in $\mathbb{Q}$, and let $\beta = \sqrt[3]{5 + \sqrt{7}}$. Recall that a polynomial is *monic* if the leading coefficient is 1.

**(a)** Find the unique monic polynomial $g \in \mathbb{Q}[x]$ of minimal degree such that $g(\beta) = 0$.

**(b)** Consider the function

$$\psi : F[x] \to E \quad \text{defined by} \quad \psi(f) = f(\beta);$$

this function is known as the *evaluation map*.
Show that $\psi(f_1) = \psi(f_2)$ if and only if $f_1 - f_2$ is divisible by $g$.